

# Complete Zeldovich approximation

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## ABSTRACT

We have developed a generalization of the Zeldovich approximation (ZA) that is exact in a wide variety of situations, including planar, spherical and cylindrical symmetries. We have shown that this generalization, that we call complete Zeldovich approximation (CZA), is exact to second order at an arbitrary point within any field. For gaussian fields, the third order error have been obtained and shown to be very small. For statistical purposes, the CZA leads to results exact to the third order.

*Subject headings:* cosmology: theory — cosmology: large-scale structure of the Universe — gravitation

Developing a simple analytical approximation that accounts accurately for the non-linear evolution of density fields seems to be a rather interesting task. To reconstruct with some accuracy the initial conditions from present velocity field in the line of sight one such approximation is needed. To obtain the statistical properties of the present field in terms of those for the initial one; the non-linear corrections to the microwave background or to the Gunn-Peterson effect, an approximation of this kind is highly convenient. An approximation that has been widely used to this purposes is Zeldovich approximation (ZA), where the density fluctuation,  $\delta$ , is a unique function of the proper values,  $\lambda_i$ , of the linearly calculated local deformation tensor  $\frac{\partial \vec{u}}{\partial \vec{x}}$  ( $\vec{u}$  being the peculiar velocity field).

$$(1 + \delta)^{-1} = \prod_{i=1}^3 (1 - \lambda_i) \quad (1)$$

This approximation give generally rather good results and it is particularly convenient for deriving the statistical properties of the present field, since we only need the statistical properties of the  $\lambda_j$  in the initial field. In this approximation the local deformation tensor is that given by the linear theory. So, although it is a good approximation to all orders, it is exact only to first order. On the other hand, in the Lagrangian perturbative development (LPD) the deformation tensor is formally exact (Bouchet et al. 1995), but it is not a unique function of the  $\lambda_i$ . Within this context, the question easily arise as to whether is possible to find an approximation depending only on the  $\lambda_i$  and substantially more accurate than ZA. Reisneger & Miralda-Escude (1995) considered an extension of ZA (EZA) that is exact for planar, spherical and cylindrical symmetry. However, although this approximation gives usually better results than ZA, it is

not fully consistent. Hui and Bertschinger (1996) developed an approximation (LTA) that is exact for any fluid element such that the orientation and axis ratios of the gravitational and velocity equipotentials are constant along its trajectory. This include planar, spherical and cylindrical symmetries. The problem with this approximation, which gives excellent results, is its complexity, that makes very difficult to determine the explicit dependence on the  $\lambda_i$ . The purpose of this letter is to present the explicit dependence of  $\delta$  as a function of the  $\lambda_i$  for the most accurate approximation that is a unique function of these quantities. This approximation we call complete Zeldovich approximation (CZA). Like EZA and LTA, CZA is exact for planar ( $\lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0$ ), spherical ( $\lambda_1 = \lambda_2 = \lambda_3$ ) and cylindrical ( $\lambda_1 = \lambda_2, \lambda_3 = 0$ ) symmetries, but, unlike those approximations, it is also exact in another wide variety of cases that essentially correspond to the collapse of an initial top-hat elliptical density fluctuation. We have shown that CZA is exact to second order and computed the third order error, which is very small. We have also shown that the exact evolution at an arbitrary point may be expressed in terms of a simple extension of the CZA containing some additional variables (the CZA being recovered when these variables are set equal to zero). All this is shown in detail in an accompanying paper; here we shall only present the CZA and comment upon its meaning and derivation.

Approximations depending only on the  $\lambda_i$  are usually called local. We shall retain this convention, but it must be noted that the  $\lambda_i$ , although defined locally, are non-locally generated. The sum of the  $\lambda_i$ , which is equal to the linear density perturbation,  $\delta_L$ , can take any value at a given point, regardless of the values taken at other points. To obtain the  $\lambda_i$ , however, we must integrate the continuity equation, so the  $\lambda_i$  at a given point depends on the whole field  $\delta_L(\vec{x})$ . Another way to point out the non-local character of the  $\lambda_i$  is by noticing that the quantities  $\lambda_i - \frac{\delta_L}{3}$  are generated by the action of the linearly calculated local tidal field which, obviously, depends on the whole field  $\delta_L(\vec{x})$ . Keeping this in mind, it is not surprising that although knowing  $\delta_L$  at a point at some initial time let us to obtain the evolution only to first order, knowing the  $\lambda_i$  let us to obtain the evolution to second order. We shall now describe the steps we have followed to obtain the CZA. We choose the ansatz:

$$(1 + \delta)^{-1} = \prod_{i=1}^3 (1 - r_i(\lambda) \lambda_i), \quad (2)$$

where  $r_i(\lambda)$  are certain functions of the  $\lambda_i$ . Within this ansatz, ZA corresponds to the zeroth order approximation for  $r_i$  ( $r_i = 1$ ). It is interesting to note that, although independently developed this ansatz is similar to that chosen by the authors of the EZA, except for the fact that they assumed all  $r_i$  to be equal; an assumption that is incompatible with exactness to second order.

To determine the functions  $r_i$  we use the constraints imposed on them by: considerations about the symmetry of  $\lambda_i$  with respect to permutations of the indexes; the fact that for planar symmetry (2) must be exact with  $r_i = 1$ ; the form of the exact solution for spherical collapse and compatibility of the form of  $r_i$  with dynamical equations. These conditions determine  $r_i$  uniquely. Let us comment them in more detail.

Rotational invariance imply that the  $r_i$  must reduce to each other through permutations of the indexes. So, they all derive from the same function,  $r(\vec{u})$ .

$$r_i(\vec{\lambda}) = r(\vec{u})|_{\vec{u}=(\lambda_i, \lambda_j, \lambda_k)} \quad (3)$$

Furthermore, rotational symmetry in the plane perpendicular to the  $i$ -th proper axis imply that  $\lambda_j, \lambda_k$  must enter symmetrically in this expression. So,  $r(\vec{u})$  must be symmetric with respect to its second and third arguments. We now assume that a series expansion of  $r(\vec{u})$  in powers of the  $u_i$  exist. We shall see later that this series converges for all relevant  $\vec{u}$ . The symmetry considerations we have just mentioned, imply that this series can only contain terms of the form:

$$r(u_1, u_2, u_3) = 1 + \sum_{l, m, n=0}^{\infty} C_{l, m, n}^p (u_2 + u_3)^l (u_2 - u_3)^{2n} u_1^m$$

$$p \equiv l + 2n + m, \quad (4)$$

where  $C_{l, m, n}^p$  are the coefficients of the  $p$ -th order terms. Noting that in the planar case ( $\lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0$ ) expression (2) is exact with  $r_i = 1$ , it is clear that expansion (4) cannot contain terms with  $m \neq 0$  and  $l = n = 0$ . So, in our notation we must have:

$$C_{0, m, 0}^p = 0 \quad (5)$$

Hence ,there are  $2p - 1$  terms of order  $p$ .

For spherical collapse, both the actual density fluctuation,  $\delta_{sp}$ , and its linear value,  $\delta_L$ , may be expressed exactly as a parametric function of time (Peebles 1980). From these expressions we have derived an expression for  $\delta$  as an explicit function of  $\delta_L$ :

$$1 + \delta_{sp} = \left( 1 - r_{sp}(\delta_L) \frac{\delta_L}{3} \right)^{-3}$$

$$r_{sp}(\delta_L) = 1 + f_1(\theta) \frac{\delta_L}{7} + f_2(\theta) \frac{23}{567} \delta_L^2 + f_3(\theta) \frac{13}{900} \delta_L^3 + f_4(\theta) 5.86 \times 10^{-3} \delta_L^4 + f_5(\theta) 2.55 \times 10^{-3} \delta_L^5 + R_{sp}(\delta_L)$$

$$R_{sp}(\delta_L) = f_6(\theta) 2.58 \times 10^{-3} \delta_L^5 \left( \frac{1}{1 - \frac{\delta_L}{2.065}} - 1 \right) + E$$

$$|E| < 2 \times 10^{-3} \text{ for } \delta_L \leq 1.57; \quad E \propto O(\delta_L^6), \quad (6)$$

where  $\theta$  stands for all cosmological parameters. For a flat Friedman model all  $f_i(\theta)$  are exactly equal to one. For a general Friedman model the dependence on  $\Omega$  (the density in units of the critical one) is very mild. When  $\Omega > 1/20$  the following is a rather good approximation:

$$f_i(\Omega) = \Omega^{2i/63} \quad (7)$$

Comparing expressions (6) and (2) and noting that for spherical symmetry  $\lambda_1 = \lambda_2 = \lambda_3 = \frac{\delta_L}{3}$ , we obtain the following constraint on  $r(u)$ :

$$r_{sp}(\delta_L) = r \left( \frac{\delta_L}{3}, \frac{\delta_L}{3}, \frac{\delta_L}{3} \right) \quad (8)$$

This relationship imply that the coefficients of order  $p$ ,  $C_{i,m,n}^p$ , must satisfy just one equation. So ,for  $p$  larger than one the coefficients are underdetermined. However, the value of  $\delta$  given by (2) must satisfy the dynamical equations (Peebles 1980):

$$\frac{d\vec{v}}{dt} + \vec{v} \frac{\dot{a}}{a} = -\frac{\nabla\phi}{a}; \quad \phi(\vec{x}) = G \int \frac{d^3x' \delta(x')}{|x' - x|}$$

$$\vec{v} = a\vec{u}; \quad \vec{u} \equiv \dot{\vec{x}}; \quad r_i(\lambda)\lambda_i = -\frac{\partial \dot{x}_i(t, q_i)}{\partial q_i} \quad (9)$$

where  $x_i$  are Eulerian comoving coordinates and  $x_i(t, q_i)$  are the Eulerian coodinates of a particle with Lagrangian coordinates  $q$  (the last relationship holds in the local proper system). Note that the continuity equation is automatically satisfied by CZA. From these equations one may see that terms of order  $p$  in  $r$  imply the existence of certain terms of order  $p + 1$ . This recursive scheme, together with expresion (8) determine completely all coefficients. Their computation, which is not trivial, is given in detail in the accompanying paper. Here we simply give the result and comment it. It is interesting to note that the process we have followed to determine the coefficients essentially amounts to analytically continue the  $r_i(\vec{\lambda})$  known in the planar and spherical case in a manner consistent with equations (9). The situations described exactly by the CZA are those where the local deformation tensor (whose proper values are  $r_i(\vec{\lambda})\lambda_i$ ) are everywhere the same (or, at least at all fluid elements affecting each other's evolution). This must be so

because it is true for the planar and spherical case and preserved by the continuing procedure. For a flat Friedman universe, we have found for  $r_i$  (in the general case terms of order  $p$  should be multiplied by  $f_p(\theta)$ ):

$$\begin{aligned} r_i(\lambda_i, \lambda_j, \lambda_k) = & 1 + \frac{3}{14}(\lambda_j + \lambda_k) + \frac{18}{245}(\lambda_j + \lambda_k)^2 + \frac{157}{4410}(\lambda_j + \lambda_k)\lambda_i + \frac{3}{245}(\lambda_j - \lambda_k)^2 + 0.03371(\lambda_j + \lambda_k)^3 \\ & + 1.63 \times 10^{-2}(\lambda_j + \lambda_k)^2\lambda_i + 2.75 \times 10^{-2}(\lambda_j + \lambda_k)\lambda_i^2 + 10^{-3}(\lambda_j - \lambda_k)^2\lambda_i + 1.2 \times 10^{-2}(\lambda_j - \lambda_k)^2(\lambda_j + \lambda_k) \\ & + 1.94 \times 10^{-2}(\lambda_j + \lambda_k)^4 + 9.4 \times 10^{-3}(\lambda_j + \lambda_k)^3\lambda_i + 1.58 \times 10^{-2}(\lambda_j + \lambda_k)^2\lambda_i^2 + 1.3 \times 10^{-2}(\lambda_j + \lambda_k)\lambda_i^3 \\ & + 4.3 \times 10^{-3}(\lambda_j - \lambda_k)^4 + 8.4 \times 10^{-3}(\lambda_j - \lambda_k)^2(\lambda_j + \lambda_k)^2 + 7.2 \times 10^{-4}(\lambda_j - \lambda_k)^2(\lambda_j + \lambda_k)\lambda_i + R(\lambda_i, \lambda_j, \lambda_k) \end{aligned} \quad (10)$$

This messy expression includes explicitly terms up to the fourth order although for more purposes using up to the quadratic term is enough. Terms of order larger than four are usually very small. However, in the rare cases when they are of some relevance many orders contribute roughly equally. So, it is not convenient to include higher order terms explicitly. Instead, we approximate all those terms by a single term,  $R$ , that we shall latter give. Expression (2) with  $r_i$  given by (10) give the exact evolution of  $\delta$  in a field where the local deformation tensor is everywhere given by  $r_i(\vec{\lambda})\lambda_i$  at a time when the linear one is  $\lambda_i$ . It is clear that this is the situation within a top-hat cylindrical fluctuation. So, we may use this case (with  $\Omega = 1$ ) to check the correctness of the continuing process. In this case we have  $\lambda_1 = 0$ ,  $\lambda_2 = \lambda_3 = \frac{\delta_L}{2}$ . So, expressions (2) and (10) lead to:

$$\begin{aligned} (1 + \delta_{\text{cyl}}) &= \left(1 - r_{\text{cyl}}(\delta_L)\frac{\delta_L}{2}\right)^2 ; \quad r_{\text{cyl}} \equiv r\left(\frac{\delta_L}{2}, \frac{\delta_L}{2}, 0\right) \\ &= 1 + \frac{3}{28}\delta_L + \frac{107}{3528}\delta_L^2 + 1.135 \times 10^{-2}\delta_L^3 + 4.5 \times 10^{-3}\delta_L^4 + R_{\text{cyl}}(\delta_L) \end{aligned} \quad (11)$$

On the other hand, we have obtained  $r_{\text{cyl}}(\delta_L)$  through direct accurate numerical integration (the error of  $r_{\text{cyl}} < 10^{-5}$ ) and fitted the coefficients in the expansion. These coefficients agree with the predictions (those in (11)) well within the fitting errors (0.1% ,0.4% ,3% ,10% for coefficients from the first to the fourth). We have also used the numerical results to fit an approximate expression for  $R_{\text{cyl}}(\delta_L)$  and find:

$$\begin{aligned} R_{\text{cyl}} &= 2.2 \times 10^{-3}\delta_L^5 \left(1 - \frac{\delta_L}{2.06}\right)^{-1} + E \\ |E| &< 5 \times 10^{-3} \end{aligned} \quad (12)$$

We may now obtain an expression for  $R(\lambda)$  (see expression (10)) demanding that the exact result be obtained in the planar and spherical cases and that it reduces to a good approximation to  $R_{\text{cyl}}$  in the cylindrical case:

$$\begin{aligned}
R(\lambda_i, \lambda_j, \lambda_k) &= \left[ 1 - 9 \left( \lambda_i - \frac{\lambda_j + \lambda_k}{2} \right) \left( 1 - \frac{\lambda_i + \lambda_j + \lambda_k}{1.3} \right) \right] \\
&\times \left( V_{\text{sp}}(\lambda_i + \lambda_j + \lambda_k) - V_{\text{sp}}(\lambda_i) + V_{\text{sp}} \left( \frac{\lambda_j + \lambda_k}{2} \right) \right) + E \\
V_{\text{sp}}(x) &\equiv r_{\text{sp}}(x) - \left( 1 + \frac{x}{7} + \frac{23}{567}x^2 + \frac{13}{900}x^3 + 5.86 \times 10^{-3}x^4 \right) \\
&= 2.58 \times 10^{-3}x^5 \left( 1 - \frac{x}{2.06} \right)^{-1} + E_2; \quad E_2 < 2.3 \times 10^{-3} \quad \text{for } x < 1.57
\end{aligned} \tag{13}$$

This expression corresponds to  $\Omega = 1$  and may be immediately generalized for an arbitrary cosmological model. The maximum error of expression (10) with  $R(\lambda)$  given by (13) is  $6 \times 10^{-3}$  being usually quite smaller. For general values of the  $\lambda_i$  the situation described exactly by expressions (2) and (10) is more complex than for the three peculiar cases already considered. However, for all practical purposes a simple generalization of these cases, namely, a top-hat initial ellipsoidal fluctuation, may be considered to be described exactly by those expressions. In fact, it may be shown that the intrinsic error of  $r_i(\vec{\lambda})$  in these situations as given by (10) ( $\approx 3 \times 10^{-3}$  at the time of collapse) is smaller than the error of this expression (due only to  $R(\lambda)$ ).

In the top-hat spherical and cylindrical cases the deformation tensor for outside matter is different from that for matter within. The same is true for the top-hat elliptical case. However, in this case, unlike in the former the outside matter is relevant to the evolution of the matter within. Expression (10) accounts exactly to the second order (and a small error to higher orders) for the small contribution of the outside matter to the tidal field within the ellipsoid.

To check the accuracy of the LTA approximation, Hui and Bertschinger (1996) considered the collapse of a top-hat initial fluctuation with axial ratios 1:1.25:1.5 and represented (in their fig.2) the evolution of the axis predicted by this approximation and that predicted by an approximation (that they called exact) that neglects the effect of outside matter. The ellipsoid generates a linear growing mode for the velocity field with associated values given by:  $\lambda_1 = 0.2576a$ ;  $\lambda_2 = 0.3233a$ ;  $\lambda_3 = 0.4191a$ . (label “1” corresponding to the largest axis). As we have said before, expressions (2) and (10) may be considered exact in this case, giving for the evolution of the axis:

$$x_i = w_i a (1 - r_i(\vec{\lambda}) \lambda_i)$$

where  $a$  is the expansion factor of the universe, and  $w_i$  are the axis ratios. This is represented in figure 1 along with the predictions of an approximation where all  $r_i$  are set equal to their symmetrized value  $\left(\frac{r_1(\lambda)+r_2(\lambda)+r_3(\lambda)}{3}\right)$ . This approximation must be very close to EZA (Reisenegger and Miralda-Escude 1995). It may be shown that by symmetrizing the effect of the outside matter is neglected. This may be checked by noting that this approximation is barely distinguishable from “exact”. The LTA and CZA are indistinguishable (within a 0.2% ) up to a  $\sim 1.2$ , differing very little up to the collapse, that takes place at  $a = 1.584$  for CZA and  $a = 1.613$  for LTA. Note that the evolution of the difference between the values of the axes given by these approximations is quantitatively the same as that between the exact solution and the LTA for an elliptical cloud in empty space (see their fig. 3), such as would be expected if CZA were exact. In fact, it may be shown that the error of the value of  $a$  at collapse given by CZA is at most 0.3% . The agreement between LTA and CZA for  $a < 1.2$  is so complete that if we had an explicit expression (like (10)) for the former approximation, the first order coefficient should be very close (within 3%) to that for the CZA and those of second order should not differ much. This imply that most likely LTA is exact to second order and that it accounts for the effect of outside matter (both facts are related), hence being more accurate than “exact” and ECZ (which can not be exact to second order).

So far we have considered the situations described exactly by CZA. However we are mostly interested in the performance of this approximation at an arbitrary point. There is no obvious reason to expect an approximation determined by the above considerations to be the best at a random point. To see that this is actually so we first write  $\delta$  at such point in the form:

$$1 + \delta = \prod_i \left( 1 - r_i \lambda_i + \frac{3}{14} \delta_L x_i + 8.46 \times 10^{-2} \delta_L^2 x_i + 8.34 \times 10^{-2} \left[ \frac{x_1^2 + x_2^2 + x_3^2}{3} \right] \delta_L + \dots \right)^{-1}$$

$$\approx 1 + \sum_i \lambda_i + \frac{10}{7} (\lambda_1 \lambda_2 + \dots) + \left[ \frac{3}{14} \delta_L + 8.46 \times 10^{-2} \delta_L^2 \right] \sum x_i \quad (14)$$

where the  $x_i$  are certain variables defined by the action of some integral operator on  $\delta_L(\vec{x})$  and that cannot be reduced to functions of the  $\lambda_i$ . This expression, that in full would contain more variables, is formally exact, like the LPD (Bouchet et al 1995). But here for each order we have separated the part depending on the  $\lambda_i$ , that goes into  $r_i(\lambda)\lambda_i$ . We may use (14) and the probability distribution of the  $\lambda_i$  at points with a fixed value of  $\delta_L$  to obtain the statistical properties of  $\delta$  at these points. By comparing what we find with the exact results found by Bernardeau (1994) for these statistical quantities we find:

$$\sum x_i = 0; \quad \langle x_i \rangle_\lambda = C \left( \lambda_i - \frac{\delta_L}{3} \right); \quad \langle x_i^2 \rangle_\lambda = \langle x_i \rangle_\lambda^2 + \frac{2C}{9} (1 - C) \sigma^2$$



$$C = \frac{3}{2}6.4(C_{\text{exp}} - 0.0544) \quad (15)$$

The first result is valid at every point and could be derived in other ways ,for example, by comparing (14) with the LPD. The other results, which are of statistical character, give the mean and mean quadratic value of  $x_i$  over points with a fixed value of the  $\lambda_i$ .  $C_{\text{exp}}$  is a spectral constant defined in the last reference.

In the context of expression (14) the CZA, as we have defined it, may be characterized by the neglect of the  $x_i$ . This approximation is the same for all fields. However, we could consider a CZA specific for each spectrum (for Gaussian fields) simply by inserting in (14) in the place of  $x_i$  ,  $x_i^2$ , their mean values given in (15) with the value of  $C_{\text{exp}}$  corresponding to that spectrum. This approximation give exactly to third order the moments of  $\delta$  over points with fixed  $\lambda_i$  values. Hence, it gives the one point statistics exactly to third order. For smooth field (the case considered here)  $C_{\text{exp}}$  lies between 0.053 and 0.061 for most interesting spectrums. So the general CZA (which is exact to third order for  $C_{\text{exp}} = 0.0544$ ) imply a very small error to third order. We have stimated the error of the CZA by computing to third order (the first non-vanishing) the RMS fluctuation of the value of  $\delta$  over points with fixed  $\lambda_i$  values. We found:

$$\langle(\delta - \langle\delta\rangle_\lambda)^2\rangle_\lambda^{1/2} = \left[ \left(\frac{3}{14}\right)^2 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (\lambda_1\lambda_2 + \dots)\frac{1}{\gamma\sigma^2} + 3.1 \right]^{1/2} \gamma\sigma^2\delta_L$$

$$\gamma \leq \frac{30}{13}(C_{\text{exp}} - 0.0471)$$

The fact that at every point the sum of the  $x_i$  vanishes imply (see (14)) that CZA is exact to second order. As we have seen, this is most likely to be also true for the LTA, but it is not true for the EZA.

Both the ZA and the CZA are unique functions of the  $\lambda_i$  so, one might wonder where is what makes the latter exact to second order. The answer is that in the CZA we use for the proper values of the deformation tensor  $r_i(\vec{\lambda})\lambda_i$  which is exact to second order, rather than  $\lambda_i$ . It must be noted however that the velocity field is not given to second order in an explicit manner. The equation  $\nabla_q \vec{u} = -\sum_i r_i \lambda_i$  is exact to second order but, to obtain the velocity field, we must integrate it.

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This manuscript was prepared with the AAS L<sup>A</sup>T<sub>E</sub>X macros v4.0.

**Figure captions:**

Figure 1: Evolution of the axis lengths for a homogeneous ellipsoid embedded in an expanding Universe. Initial axial ratios are 1:1.25:1.50. The solid line corresponds to CZA, which is practically exact; and the dashed line corresponds to an approximation that makes all  $r_i$  equal to their symmetrized value.